Stability and convergence of a higher order rational difference Equation

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Abstract

In this paper the asymptotic stability of equilibria and periodic points of the following higher order rational difference Equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma x_n x_{n-1} \dots x_{n-k}},\ k \geq 1,\ n = 0, 1, \dots$$

is studied where the parameters α, β, γ are positive real numbers, and the initial conditions $x_{-k}, \dots, x_0$ are given arbitrary real numbers. The forbidden set of this equation is found and then, the order reduction method is used to facilitate the analysis of its asymptotic dynamics.

Keywords: Difference Equation; Equilibrium point; periodic point; convergence; semiconjugacy

1 Introduction and preliminaries

Difference equations may appear as solutions of various phenomena or as a discretized system of delay or non-delay differential equations. They are very important in both theory and applications; for applications in biology (see [2]), economics (see [8, 5]), medical sciences (see [9]), military sciences (see [4]). The main goal in the study of difference equations is to understand the asymptotic behavior of solutions rather than trying to find an explicit formula for solutions. This basically is not only because the explicit solutions are hard to find but also the explicit solutions may still represent a complex dynamics and they yet may require a qualitative analysis to understand their dynamics.

The ratio of any two polynomials of a recursive sequence is called a rational difference equation. Rational difference equations are important as practical classes of difference equations. Most of the works in the literature of rational difference equations

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have treated the first and second order difference equations. For second order rational difference equations we refer the reader to the monograph of Kullenovic and Ladas ([7]). In this paper we study the dynamics of the following $(k+1)$-order rational difference equation

$$x_{n+1} \quad = \quad \frac{\alpha x_{n-k}}{\beta + \gamma x_n x_{n-1} \ldots x_{n-k}}, \qquad n = 0, 1, \ldots \tag{1}$$

where $k \geq 1$ and the parameters α, β, γ are positive. We allow the initial conditions $x_{-k}, x_{-k+1}, \ldots x_0$ to take any arbitrary value out of the forbidden set of equation (1). Let I be some interval of positive real numbers and $f : I^{k+1} \to I$ be a continuously differentiable function. Then for every initial conditions $(x_{-k}, x_{-k+1}, \ldots, x_0) \in I^{k+1}$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$. The point $\overline{x} \in I$ is called an equilibrium of equation (2) (or simply an equilibrium of f) if $f(\overline{x}, \overline{x}, \ldots, \overline{x}) = \overline{x}$, i.e., $x_n = \overline{x}$ for all $n \geq 0$ (such a solution is also called a trivial solution). The point $(c_1, \ldots, c_{k+1})$ is called a $(k+1)$-cycle if $x_{(k+1)n-i} = c_{k+1-i}$ for all $i = 0, 1, \ldots, k$. In this case we say that $\{x_n\}$ is periodic with period $(k+1)$. The linearized equation associated with equation (2) about the equilibrium $\overline{x}$ is

$$z_{n+1} = \sum_{i=0}^{k} \frac{\partial f}{\partial u_i}(\overline{x}, \ldots \overline{x}) z_{n-i},$$

and its corresponding characteristic equation is defined by

$$\lambda^{k+1} - \sum_{i=0}^{k} \frac{\partial f}{\partial u_i}(\overline{x}, \ldots \overline{x}) \lambda^{k-i} = 0. \tag{3}$$

An equilibrium $\overline{x}$ of equation(2) is called locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for the solution $\{x_n\}_{n=-k}^{\infty}$ of equation (2) with $|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \ldots + |x_0 - \overline{x}| < \delta$ we have $|x_n - \overline{x}| < \epsilon$ for all $n \geq 1$. Furthermore, if there exists $\gamma > 0$ such that for the solution $\{x_n\}_{n=-k}^{\infty}$ of equation (2) with $|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \ldots + |x_0 - \overline{x}| < \gamma$ we have $\lim_{n\to\infty} x_n = \overline{x}$, then $\overline{x}$ is called locally asymptotically stable. Linearized stability theorem indicates that if all roots of equation (3) lie inside the open unit disk $|\lambda| < 1$, then the equilibrium $\overline{x}$ of equation (2) is locally asymptotically stable. If at least one of the roots of equation (3) has modulus greater than one, then the equilibrium $\overline{x}$ of equation (2) is unstable, see e.g., [6].

$\overline{x}$ is called a global attractor on an interval I if for every solution $\{x_n\}_{n=-k}^{\infty}$ of equation (2) with $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$, we have $\lim_{n\to\infty} x_n = \overline{x}$. When $\overline{x}$ is locally stable and a global attractor, we call it globally asymptotically stable. The forbidden set of equation (2) is the set of all (k+1)-tuples $(f_1, f_2, \ldots f_{k+1})$, where they can not be

taken as the initial conditions for an infinite well-defined sequence $\{x_n\}$ on its domain. In other words, the forbidden set of equation (2) is the set of all initial conditions such that arbitrary iterations of the right hand side of equation (2) are not well-defined.

The following definitions, lemma, corollary, and theorems are needed to study the global behavior of solutions of equation (1).

Definition 1.1. (for original ideas see [3, 11]) Consider equation (2) and assume that $D \subseteq \mathbb{R}^{k+1}$ is nonempty.

(i) Suppose that $(x_1, \ldots x_{k+1}) \in \mathbb{R}^{k+1}$. Then $\|x\| = \max\{x_1, \ldots, x_{k+1}\}$ denotes the sup-norm of x. Also, if $\overline{x}$ is the equilibrium of f then $\overline{X} = (\overline{x}, \ldots, \overline{x}) \in \mathbb{R}^{k+1}$ is an equilibrium of its *vectorization* that is

$$V_f(x_1, \ldots x_{k+1}) = (f(x_1, \ldots x_{k+1}), x_1, \ldots, x_k).$$

(ii) If there is a non-constant function $H \in C(D, \mathbb{R})$ such that $H \circ V_f = \phi \circ H$ on D for some $\phi \in C(H(D), \mathbb{R})$ then V_f is also called a (D, H, ϕ)-*semiconjugate* of $\mathbb{R}^{k+1}$. The mapping H is called a *link map* and ϕ is called the *factor map.* For each $t \in H(D)$, the level set $H^{-1}(t) \cap D$, abbreviated as H_t^{-1} is called a *fiber* of H in D.

(iii) A continuous mapping $h : D \to \mathbb{R}$ is said to be *bending* at a point $x \in D$ if x is not an isolated point in D and $x \notin [h^{-1}(h(x))]^\circ \cap D$, where S° denotes the interior of S.

(iv) An equilibrium $\overline{x}$ of equation (2) is exponentially stable under f relative to some nontrivial interval I containing $\overline{x}$ if there is $\gamma \in (0, 1)$ such that for every solution $\{x_n\}$ of equation (2) with initial values $x_{-k}, \ldots, x_0 \in I$ we have for all $n \geq 1$ that $|x_n - \overline{x}| < c\gamma^n$ where $c = c(x_0, \ldots x_{-k})$ is independent of n.

(v) An equilibrium $\overline{x}$ of a map $f : \mathbb{R} \to \mathbb{R}$ is semistable (from the right) if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < x_0 - \overline{x} < \delta$ then $|f^n(x_0) - \overline{x}| < \epsilon$ for all $n \geq 1$. If in addition, $\lim_{n\to\infty} f^n(x_0) = \overline{x}$ whenever $0 < x_0 - \overline{x} < \gamma$ for some $\gamma > 0$, then $\overline{x}$ is said to be semiasymptotically stable (from the right). Semistability (semiasymptotic stability) from the left is defined analogously.

Theorem 1.1. *(see [11]). Let V_f is a (D, H, ϕ)-semiconjugate map and $\overline{x} \in D$ is an equilibrium of V_f.*

(i) *$\overline{t} = H(\overline{x})$ is an equilibrium of ϕ.*

(ii) *(Boundedness). Assume that $|H(x)| \to \infty$ as $\|x\| \to \infty$. If the sequence $\{\phi^n(t_0)\}$ is bounded for some $t_0 \in H(D)$, then the sequence $\{V_f^n(x_0)\}$ with $x_0 \in H_{t_0}^{-1}$ is bounded.*

(iii) *(Stability and instability). Assume that H is bending at $\overline{x}$. If we set $\overline{t} = H(\overline{x})$ then*

(a) *If $\overline{x}$ is stable (asymptotically stable) under V_f, then $\overline{t}$ is a stable (asymptotically stable) equilibrium of ϕ.*

(b) *If $\overline{t}$ is unstable under ϕ, then $\overline{x}$ is unstable under V_f.*

(iv) *(Attractivity of invariant fibers). Let $\overline{t} \in I$ be an isolated equilibrium of ϕ which attracts all points in I. If $x_0 \in D \cap H^{-1}(I)$ and $\{V_f^n(x_0)\}$ is bounded, then $\{V_f^n(x_0)\}$ converges to the invariant fiber $H^{-1}(\overline{t})$.*

Theorem 1.2. *(see [3]). Let $\overline{x}$ be an equilibrium of $f : \mathbb{R} \to \mathbb{R}$, $f'(\overline{x}) = 1$, and $f''(\overline{x}) \neq 0$. Then $\overline{x}$ is semiasymptotically stable from the right (left) if $f''(\overline{x}) < 0$ ($f''(\overline{x}) > 0$).*

2 The Forbidden Set

Consider equation (1). If $\alpha = 0$, the solution is trivial. If $\gamma = 0$ then equation (1) reduces to a linear equation. If $\beta = 0$ then the solution is periodic with period $k+1$. Thus, we assume that all the parameters are nonzero. A change of variables $x_n = \sqrt[k+1]{\frac{\beta}{\alpha}} y_n$ followed by the change $y_n = x_n$ reduces equation (1) to

$$x_{n+1} = \frac{cx_{n-k}}{1 + x_n x_{n-1} \ldots x_{n-k}}, \quad n = 0, 1, \ldots \tag{4}$$

where $c = \frac{\alpha}{\beta} \sqrt[k+1]{\frac{\beta}{\alpha}}$. Hence, , we consider equation (4) instead of equation (1), hereafter. Now we are ready to obtain the forbidden set of equation (4).

Theorem 2.1. *Consider equation (4). Assume that $\mathscr{F}$ is the forbidden set for this equation. Then*

$$\mathscr{F} = \bigcup_{m=0}^{\infty} \left\{ (f_1, f_2, \ldots, f_{k+1}) : f_1 f_2 \ldots f_{k+1} = \frac{-1}{\sum_{i=0}^{m} c^i} \right\}$$

Proof. Assume the initial conditions $x_{-k}, x_{-k+1}, \ldots x_0$ satisfy

$$x_{-k} x_{-k+1} \ldots x_0 = \frac{-1}{\sum_{i=0}^{m} c^i}$$

for some $m \in \mathbb{N} \cup \{0\}$. Then equation (4) implies that

$$x_1 x_0 \ldots x_{1-k} = \frac{cx_0 x_{-1} \ldots x_{1-k} x_{-k}}{1 + x_0 \ldots x_{-k}} = \frac{c\left(\frac{-1}{\sum_{i=0}^{m} c^i}\right)}{1 - \sum_{i=0}^{m} c^i} = \frac{-1}{\sum_{i=0}^{m-1} c^i}.$$

Therefore, we obtain from equation (4) that

$$x_2 x_1 \ldots x_{2-k} = \frac{cx_1 x_0 \ldots x_{2-k} x_{1-k}}{1 + x_1 \ldots x_{1-k}} = \frac{c\left(\frac{-1}{\sum_{i=0}^{m-1} c^i}\right)}{1 - \sum_{i=0}^{m-1} c^i} = \frac{-1}{\sum_{i=0}^{m-2} c^i}.$$

Continuing in this fashion by induction we obtain that

$$x_m x_{m-1} \dots x_{m-k} = \frac{-1}{\sum_{i=0}^{m-m} c^i} = -1.$$

As a result the iteration process stops at x_{m+1}. Conversely, assume iteration process stops at some point x_{n_0+1}, i.e., $x_{n_0} x_{n_0-1} \dots x_{n_0-k} = -1$. Then equation (4) implies that

$$x_{n_0-1} x_{n_0-2} \dots x_{n_0-k-1} = \frac{-1}{1+c}.$$

Again using this fact and equation (4) we obtain that

$$x_{n_0-2} x_{n_0-3} \dots x_{n_0-k-2} = \frac{-1}{1+c+c^2}.$$

Continuing in this manner we get

$$x_0 x_{-1} \dots x_{-k} = \frac{-1}{\sum_{i=0}^{n_0} c^i}$$

or equivalently $(x_{-k}, x_{-k+1}, \dots x_0) \in \mathscr{F}$. This completes our inductive proof. □

3 Linearized Stability

The first step to understand of the dynamics is to find the equilibria and find their stability type. In this section we investigate the local asymptotic stability of the equilibria of equation (4). Simple calculations show that origin is always an equilibrium for equation (4) and if $c > 1$ then it has a second equilibrium $\overline{x} = \sqrt[k+1]{c-1}$. The following theorem deals with the local asymptotic stability of the equilibria. Note that throughout the rest of this paper, the initial conditions are assumed to be taken out of the forbidden set $\mathscr{F}$.

Theorem 3.1. *Consider equation (4). Then,*

(a) *origin is locally asymptotically stable for $c < 1$, while it is unstable when $c > 1$.*

(b) *For the case of $c > 1$, the positive equilibrium $\overline{x} = \sqrt[k+1]{c-1}$ is stable.*

Proof. For part (a), the linearized equation associated with equation (4) about origin is $z_{n+1} - cz_{n-k} = 0,\ n \geq 0$. Therefore, the corresponding characteristic equation is $\lambda^{k+1} - c = 0$, i.e., $\lambda = \sqrt[k+1]{c}$. Since the origin is locally asymptotically stable if $c < 1$ and is unstable if $c > 1$.

(b) Simple calculations show that the linearized equation associated with equation (4) about the positive equilibrium $\overline{x} = \sqrt[k+2]{c-1}$ is

$$z_{n+1} + \frac{c-1}{c} z_n + \frac{c-1}{c} z_{n-1} + \dots + \frac{c-1}{c} z_{n-k+1} - \frac{1}{c} z_{n-k} = 0, \quad n = 0, 1, \dots$$

Therefore the corresponding characteristic equation is

$$\lambda^{k+1} + \frac{c-1}{c}\lambda^k + \ldots + \frac{c-1}{c}\lambda - \frac{1}{c} = 0, \tag{5}$$

Some algebra show that equation (5) is equivalent to

$$\frac{(\lambda^{k+1}-1)(\lambda-1/c)}{\lambda-1} = 0, \quad \lambda \neq 1, \tag{6}$$

Therefore, $\lambda = 1/c$ is one of the roots of equation (6). Since $c < 1$, by local asymptotic stability Theorem we conclude that the positive equilibrium $\overline{x} = \sqrt[k+1]{c-1}$ is unstable. The proof is complete. □

Remark 3.1. *The above theorem is sufficient to completely describe the local dynamics of the equilibrium for the case $c < 1$. However, for the case $c \leq 1$, more investigation are needed. For $c = 1$ the characteristic equation about origin has modulus equal to one, the linearization fails to analyze the asymptotic stability of origin. Thus, it may have a complex dynamics near this point. On the other hand consider equation (6) and assume that $c > 1$. equation (6) has a real root $\lambda = 1/c$ with modulus less than one but this equation has $k+1$ roots in the following form*

$$\lambda_m = \cos\left(\frac{2m\pi}{k+1}\right) + i\sin\left(\frac{2m\pi}{k+1}\right), \quad m = 0, 1, \ldots, k,$$

all with modulus equal to one. So, if $c > 1$ then linearization tells us nothing about the stability of the positive equilibrium. In the next section we discuss these cases in details.

4 Semiconjugate factorization

The main purpose of this section is to analyze the local dynamics near equilibria for the cases $c \geq 1$ as well as of the global asymptotic dynamics of the equation (4). Although the previous section, using linearization, showed that the origin is locally asymptotically stable for $c < 1$. Yet the global nature of asymptotic stability can not be inferred from linearization. Also, we saw that for $c > 1$ the linearized equation about the positive equilibrium has several roots; all with modulus equal to unity. As a result, more powerful methods are necessary in order to analyze the dynamics of equation (4). In this section, we apply semiconjugacy analysis to examine the global nature of equation (4). The idea is to reduce the order of a higher order difference equation such that the analysis of the reduced system is feasible. Then, this is helpful when this analysis can facilitate the understanding of the dynamics for the original difference equation.

Let $f(x_1, x_2, \ldots, x_{k+1}) = \frac{cx_{k+1}}{1+x_1x_2\ldots x_{k+1}}$. Consider the vectorization of f i.e.,

$$V_f(x_1, x_2, \ldots, x_{k+1}) = \left(\frac{cx_{k+1}}{1+x_1x_2\ldots x_{k+1}}, x_1, \ldots, x_k\right).$$

Set $H(x_1, x_2, \ldots, x_{k+1}) = x_1 x_2 \ldots x_{k+1}$ and note that

$$H(V_f(x_1, x_2, \ldots, x_{k+1})) = \frac{c x_1 x_2 \ldots x_{k+1}}{1 + x_1 x_2 \ldots x_{k+1}} = \frac{cH(x_1, x_2, \ldots, x_{k+1})}{1 + H(x_1, x_2, \ldots, x_{k+1})},$$

So the map H makes V_f a (D, H, ϕ)-semiconjugate map with D being the nonnegative orthant of $\mathbb{R}^{k+1}$, i.e., $D = [0, \infty)^{k+1}$, and the map

$$\phi(t) = \frac{ct}{1+t},$$

serving as the factor map on $[0, \infty)$. Therefore, equation (4) is a semiconjugae factorization of the well-known Ricatti difference equation $t_{n+1} = \phi(t_n) = \frac{ct_n}{1+t_n}$. In the sequence, we consider three cases as follow:

Case I: $0 < c < 1$; In this case we claim that

$$0 \leq x_{n+1} \leq c^{\lceil n/k+1 \rceil} \max\{x_{-k}, x_{-k+1}, \ldots, x_0\}, \quad n \geq -k,$$

where $\lceil n \rceil$ is the greatest integer which is less than or equal to n. To prove the above claim note that it is true for all $-k \leq n \leq -1$. Assume that it is true for all integers less than or equal to some integer $n(n > -1)$. Then

$$\begin{aligned} x_{n+1} &= \frac{cx_{n-k}}{1 + x_n x_{n-1} \ldots x_{n-k}} \\ &\leq cx_{n-k} \\ &\leq c^{\lceil \frac{n-k}{k+1} \rceil + 1} \max\{x_{-k}, x_{-k+1}, \ldots, x_0\} \\ &= c^{\lceil \frac{n+1}{k+1} \rceil} \max\{x_{-k}, x_{-k+1}, \ldots, x_0\}, \end{aligned}$$

and this completes our inductive proof. Therefore, in this case every positive solution of equation (4) converges exponentially to origin.

Case II: $c > 1$; In this case the factor map ϕ has a positive equilibrium $\bar{t} = c - 1$. Recall that in a Ricatti equation the positive equilibrium is globally asymptotically stable (see [7]. P79). Hence, $\bar{t}$ is globally asymptotically stable equilibrium of ϕ. Also, since the sequence $\{\phi(t_0)\}_{n=0}^{\infty}$, $t_0 \in (0, \infty)$ is bounded (this is evident by the global asymptotic stability of $\bar{t}$), Theorem 1.1*(ii)* implies that the sequence $\{V_f^n(x_0)\}_{n=0}^{\infty}$, $x_0 \in (0, \infty)^{k+1}$ is bounded. Thereby, we conclude from Theorem 2*(iv)* that the sequence $\{V_f^n(x_0)\}_{n=0}^{\infty}$ converges to the invariant fiber $H_{\bar{t}}^{-1}$ since the sequence $\{\phi(t_0)\}_{n=0}^{\infty}$ converges to $\bar{t}$. Now, consider the fiber $H_{\bar{t}}^{-1}$. If $(x_1, x_2, \ldots x_{k+1}) \in H_{\bar{t}}^{-1}$ then some simple algebra show that

$$V_f^{k+1}(x_1, x_2, \ldots, x_{k+1}) = (x_1, x_2, \ldots, x_{k+1}),$$

Therefore, every member of $H_{\bar{t}}^{-1}$ is a $(k+1)$-cycle of V_f; in other words, it is a periodic orbit of equation (4) of period $(k+1)$. Then, in this case every solution of equation (4) converges to a $(k+1)$-cycle.

Case III: $c = 1$; In this case origin is the unique equilibrium of equation (4) and ϕ. Some calculations show that

$$\phi'(0) = 1, \quad \phi''(0) = -2,$$

Therefore by Theorem 1.2 origin is semiasymptotic stable equilibrium of ϕ form the right, i.e., origin attracts the sequence $\{\phi^n(t_0)\}$ for all $t_0 \in (0, \infty)$. So, by an analysis precisely similar to that of used in the previous case we conclude, by Theorem 2*(iv)*, that the the fiber H_0^{-1} attracts every solution of equation (4) since origin attracts every solution of the sequence $\{\phi^n(t_0)\}$. Now, assume that $(x_1, x_2, \dots, x_{k+1}) \in H_0^{-1}$. Then

$$x_1 x_2 \dots x_{k+1} = 0,$$

and therefore, every solution of equation (4) either converges to origin or to a point in one of $(k+1)$ coordinate planes in $\mathbb{R}^{k+1}$, i.e., the following set

$$S = \{(x_1, x_2, \dots, x_{k+1}) | \; \exists \; 1 \le i \le k+1 \;\; \ni \;\; x_i = 0 \; \}. \tag{7}$$

elements of the attracting set S (note that $S = \cup S_i$, where S_i are the coordinate planes) are $(k+1)$-cycles of equation (4). To prove this consider $(x_1, \dots, x_i, 0, x_{i+1}, \dots, x_{k+1}) \in S$. Then, it is easy to verify that

$$V_f^{k+1}(x_1, \dots, x_i, 0, x_{i+1}, \dots, x_{k+1}) = (x_1, \dots, x_i, 0, x_{i+1}, \dots, x_{k+1}). \tag{8}$$

Consequently, every solution of equation (4) either converges to the origin or to a periodic orbit of period $(k+1)$, that is in a coordinate plane of $\mathbb{R}^{k+1}$. We now summarize the above arguments into the following theorem that is one of the main result of this paper.

Theorem 4.1. *(Basic convergence Theorem) Assume Equation (4) is given. Then,*

(i) *For* $0 < c < 1$, *every positive orbit of equation (4) converges exponentially to origin. This implies that the equation has no other periodic orbit except the trivial fixed point (i.e., origin).*

(ii) *When* $c \ge 1$, *any cycle of the equation (4) is a* $(k+1)$*-cycle and the set of all* $(k+1)$*-cycles is given by equation (7). Furthermore*

1. *For* $c = 1$, *every positive orbit of equation (4) either converges to origin or to a* $(k+1)$*-cycle.*
2. *When* $c > 1$, *every positive solution of equation (4) converges to a* $(k+1)$*-cycle.*

References

[1] E. Braverman, On a discrete model of population dynamics with impolsive harvisting or recruitment, Nonlinear Analysis, **63** (2005) 751–759.

[2] F. Brauer, C. Castillo-Chavez, Mathematical Models in Population Biology and Epidemiology, Springer, New York, 2001

[3] S. N. Elaydi, Discrete Chaos, Chapman & Hall/CRC, 2000.

[4] J. M. Epstein, The Calculus of Conventional War, Brookings inst. Studies in defence policy, Washington, DC, 1985.

[5] Ishiyama K, Saiki Y, Unstable periodic orbits and chaotic economic growth, Chaos, Solitins & Fractals, **26** (2005) 33–42.

[6] V. L. Kocic, G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Accademic Publishers, 1993.

[7] M. R. S. Kulenovic, G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman and Hall/CRC, 2002.

[8] L. Matti, Oscillations in some nonlinear economic relationships, Chaos, Solitons & Fractals, **7** (1996) 2235–45.

[9] L. A. Moye, A. S. Kapadia, Differenc equations with public health applications, Marcel Dekker, Inc. 2000.

[10] H. Sedaghat, Converges, oscillations, and chaos in a discrete model of combat, SIAM. Reveiw **44** (2002) 74–92.

[11] H. Sedaghat, Nonlinear Difference Equations: Theory with Applications to Social Science Models, Kluwer, Dordrecht, The Netherlands, 2003.

[12] H. Sedaghat, M. Shojaei, A class of second order difference equations inspired by Euler's discretization method, International Journal of Applied Mathematics & Statistics, **9** (2007) 110–123.

[13] M. Shojaei, R. Saadati, H. Adibi, Stability and periodic character of a rational third order difference equation, Chaos, Solutons & Fractals, **39** (2009) 1203–1209.